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SOME STRATIGRAPHIC CONTROL PROBLEMS

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In this paper, we are interested in lithology diffusion models applied in the field of stratigraphic basin simulations for large-scale depositional transport processes of sediments. Such models describe erosion-sedimentation processes and take into account limited weathering via nonstandard unilateral problems. Various theoretical results, illustrations, and numerical solutions are presented for the monolithologic column case. A new conservation law involved in modeling is formulated, and mathematical tools for solving the problem are described.

Key words: *stratigraphic models, limited weathering, ill-posed problems, inverse problems, degenerated parabolic–hyperbolic conservation laws.*

Introduction. New stratigraphic models recently developed by the Institut Francais du Petrole (IFP) lead to mathematical questions that are difficult to answer within the framework of ill-posed and inverse problems. The main processes in sedimentary basin evolution are the erosion, deposition mechanism of sediments, and vertical compaction. At large scales in time and space, dynamic-slope approaches are usually preferred rather than fluid flow models (see [1, 2]).

The model proposed in [1, 3–5] offers a mathematical description of the coupling of the limited weathering erosion and the nonlinear lithology diffusion models: the limited weathering process is expressed as an inequality constraint on the partial time derivative of sediment thickness and a new unknown introduced to limit the fluxes if the need arises. The compatible coupling between both models is obtained either by imposing unilateral conditions for the flux limiter and inequality constraints on the erosion rate or by looking for a maximal limiter.

The model proposed does not take into account compaction phenomena [6].

1. Modeling. Let us consider a sedimentary basin with a base Ω. Let $Q =]0, T[\times \Omega$ for any positive T.

Physical modeling is based on three assumptions.

Assumption 1. The model is weathering-limited, i.e., the erosion rate is underestimated by a given nonnegative bounded measurable function E in Q: $\partial_t h \geqslant -E$, where h is the sediment height (topography).

Assumption 2. Unilateral constraints on the outflow boundary Γ_e have the form $\lambda \partial_n h + f \geqslant 0$, $\partial_t h + E \geqslant 0$, and $(\lambda \partial_N h + f)(\partial_t h + E) = 0$, where f is a given bounded measurable function on Σ.

To reconcile Assumptions 1 and 2 with the conservative formulation, we use the following assumption:

Assumption 3. The flux of matter $\boldsymbol{q}$ is proportional to ∇h, i.e., $\boldsymbol{q} = \lambda \nabla h$ in Q, where $\lambda = \lambda(t, x)$ is a suitable duality multiplier *a priori* located in the interval $[0, 1]$.

Thus, the mathematical modeling leads to the following assertions:

— mass balance of the sediment

$$\partial_t h - \operatorname{div}(\lambda \nabla h) = 0 \qquad \text{in} \quad Q; \tag{1}$$

— boundary conditions on $\partial\Omega = \overline{\Gamma_e} \cup \overline{\Gamma_s}$:

$$-\lambda \partial_n h = f \qquad \text{on} \quad]0, T[\times \Gamma_s; \tag{2}$$

$$\lambda \partial_n h + f \geqslant 0, \quad \partial_t h + E \geqslant 0, \quad (\lambda \partial_n h + f)(\partial_t h + E) = 0 \quad \text{on} \quad]0, T[\times \Gamma_e; \tag{3}$$

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— limited weathering conditions

$$\partial_t h \geqslant -E \quad \text{in} \quad Q; \tag{4}$$

— initial conditions

$$h(0, x) = h_0 \quad \text{in} \quad \Omega. \tag{5}$$

Definition 1. A strong solution to problem (1)–(5) is a pair (λ, h) in $L^\infty(Q) \times L^2(0,T;H^1(\Omega))$ such that

$$0 \leqslant \lambda \leqslant 1 \quad \text{in } Q, \qquad h(0, x) = h_0 \quad \text{in } \Omega,$$

$$\partial_t h \in L^2(0,T;H^1(\Omega)), \qquad \partial_t h \geqslant -E \quad \text{on } \Gamma_s,$$

and $\forall v \in H^1(Q)$ and $t \in]0,T[$, the following inequality is valid:

$$\int_\Omega \partial_t h(v - \partial_t h)\, dx + \int_\Omega \lambda \nabla h \nabla (v - \partial_t h)\, dx + \int_\Gamma f(v - \partial_t h)\, d\sigma + \int_{\Gamma_s} \chi_{\mathbb{R}^+}(v + E)\, d\sigma \geqslant 0.$$

Here $\chi_{\mathbb{R}^+}(x) = 0$ if $x \geqslant 0$ and $\chi_{\mathbb{R}^+}(x) = +\infty$ if $x < 0$.

2. Strong Solution for a Preparatory Problem. We consider now a preparatory problem (1)–(3), (5) [without condition (4)].

2.1. *Existence of a Strong Solution.* The following proposition is valid.

Proposition 1. *Assume that E and f are regular functions such that $\lambda \in W^{1,\infty}(0,T;L^\infty(\Omega))$ for $0 < a \leqslant \lambda \leqslant 1$ and that there exists a nonnegative function $g \in L^2(\Omega)$ such that, for any $v \in H^1(\Omega)$, one has*

$$\int_\Omega [g - E(0)]v\, dx + \int_\Omega \lambda(0) \nabla h_0 \nabla v\, dx + \int_\Gamma f(0) v\, d\sigma = 0. \tag{6}$$

Then, there exists a unique strong solution h to the preparatory problem (1)–(3), (5).

Let us give a sketch of the proof (see [4, 7] for more details).

1. Based on the Galerkin method, one can prove that, $\forall \varepsilon, \eta > 0$, there exists a solution $h_{\varepsilon,\eta} \in L^2(0,T;H^1(\Omega))$ such that

$$\partial_t h_{\varepsilon,\eta} \in L^2(0,T;H^1(\Omega)), \qquad \partial_t^2 h_{\varepsilon,\eta} \in L^2(0,T;L^2(\Omega)),$$

$$h_{\varepsilon,\eta}(0,x) = h_{0,\eta}, \qquad \partial_t h_{\varepsilon,\eta}(0) = g_\eta - E,$$

where g_η is a nonnegative sequence of the $H^1(\Omega)$ functions, which converges to g in $L^2(\Omega)$; $h_{0,\eta}$ is a solution to (6), and $\forall v \in H^1(\Omega)$ and $t \in]0,T[$, the following equality is satisfied:

$$\varepsilon \int_\Omega \partial_t^2 h_{\varepsilon,\eta} v\, dx + \int_\Omega \partial_t h_{\varepsilon,\eta} v\, dx + \int_\Omega \lambda \nabla h_{\varepsilon,\eta} \nabla v\, dx + \int_\Gamma f v\, d\sigma - \int_{\Gamma_s} \beta_\eta(\partial_t h_{\varepsilon,\eta} + E) v\, d\sigma = 0.$$

Here $\beta_\eta(x) = -(x/\eta) I_{]-\infty,-1]} + (x/\eta)(x^2 + 2) I_{]-1,0]}$; $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.

It was proved in [4] that there exist positive constants M and M'_η (dependent on η) such that, for any t, one has

$$\|h_{\eta,\varepsilon}(t)\|_V + \|\partial_t h_{\eta,\varepsilon}(t)\|_H + \|\partial_t h_{\eta,\varepsilon}\|_{L^2(0,t;V)} + \|\eta^{-1} G[\partial_t h_{\eta,\varepsilon} + E]\|_{L^1(0,t;H)} \leqslant M(1 + \sqrt{\varepsilon}),$$

$$\|\sqrt{\varepsilon}\, \partial_t^2 h_{\eta,\varepsilon}(t)\|_H + \|\partial_t^2 h_{\eta,\varepsilon}\|_{L^2(0,t;H)} + \|\partial_t h_{\eta,\varepsilon}(t)\|_V \leqslant M'_\eta$$

$[G(x) = -\beta(x)x]$.

2. Using some compactness arguments, one verifies that $\forall \eta > 0\ \exists h_\eta \in L^2(0,T;H^1(\Omega))$ such that

$$\partial_t h_\eta \in L^2(0,T;H^1(\Omega)), \qquad \partial_t^2 h_\eta \in L^2(0,T;[H^{-1}(\Omega)]'),$$

$$h_\eta(0, x) = h_{0,\eta}, \qquad \partial_t h_\eta(0) = g_\eta - E,$$

and $\forall v \in H^1(\Omega)$ and $t \in]\,0,T\,[$, the following equality is satisfied:

$$\int_\Omega \partial_t h_\eta v\, dx + \int_\Omega \lambda \nabla h_\eta \nabla v\, dx + \int_\Gamma f v\, d\sigma - \int_{\Gamma_s} \beta_\eta(\partial_t h_\eta + E) v\, d\sigma = 0.$$

Moreover, there exists a positive constant M such that

$$\|h_\eta\|_{L^\infty(0,T;V)} + \|\eta^{-1}G[\partial_t h_\eta + E]\|_{L^1(Q)} + \|\partial_t h_\eta\|_{L^2(0,T;V)} \leqslant M$$

and $\|\partial_t^2 h_\eta\|_{L^2(0,T;V_0')} \leqslant M$. Here $V_0 = \{v \in V\colon\ v = 0 \text{ in } \Gamma_s\}$.

3. The existence of the solution is proved with the use of standard compactness and convexity arguments and weighted time integration.

4. The uniqueness of the solution is proved by monotonicity arguments.

2.2. *Limited Weathering Condition.* Let us give one of the sufficient conditions of limited weathering.

Proposition 2. *Assume that λ, E, and f satisfy the inequalities*

$$\operatorname{div}(\partial_t\lambda\nabla h) + \partial_t E - \operatorname{div}(\lambda\nabla h) \geqslant 0 \quad \textit{in } Q,$$

$$\partial_t f - \partial_t\lambda\nabla h\cdot \boldsymbol{n}_{ext} + \lambda\nabla E\boldsymbol{n}_{ext} \geqslant 0 \quad \textit{on} \quad]0,T[\times\Gamma_e.$$

Then, h satisfies the inequality $\partial_t h + E \geqslant 0$ in Q.

The proof is based on the maximum principle for the function $\partial_t h$.

Example 1. Assume that $E(t,x) = E_0$, $f(t,x) = f(x)$, and $\lambda(t,x) = \lambda(x)$, then one has $\partial_t h + E \geqslant 0$ in Q.

3. Modeling of the Parameter $\boldsymbol{\lambda}$. Let us denote by $\Lambda_{\rm ad}$ the admissible set of parameters:

$$\Lambda_{\rm ad} = \{\lambda \in L^\infty(Q)\colon \exists h,\ \text{where } (\lambda,h) \text{ is a solution of (1)–(5)}\}.$$

Example 2. Assume that $E = a \geqslant 0$, $f = 0$, and $h_0 = c > 0$. Then, for any measurable $0 \leqslant \lambda \leqslant 1$, the pair (λ, h_0) is a solution of problem (1)–(5) and $\lambda \in \Lambda_{\rm ad}$. Therefore, the problem has a nonunique solution, and other assumptions are needed to determine λ.

Model 1. *Maximal value of* λ. We assume that the value of λ is maximal in $\Lambda_{\rm ad}$, i.e.,

1) $\lambda \in \Lambda_{\rm ad}$;

2) $\forall\mu \in \Lambda_{\rm ad}$, $\mu \leqslant \lambda$ in Q.

Model 2. *Unilateral Constraint for Limited Weathering.* We assume that the weathering condition is effective for $\lambda < 1$, i.e.,

1) $\lambda \in \Lambda_{\rm ad}$;

2) $(1-\lambda)(\partial_t h + E) = 0$ in Q.

In other words, if the maximum erosion rate is not reached, then $\lambda = 1$; and conversely, if the maximum erosion rate is attained, $\lambda < 1$.

3.1. *Heuristic Approach and Ill-Posedness of the Models.* Let us give some examples.

Example 3. Assume that $E = a \geqslant 0$, $f = 0$, and $h_0 = c > 0$. Then:

1) the pair $(1, h_0)$ is the unique solution of Model 1;

2) the pair $(1, h_0)$ is a solution of Model 2.

Note, if $a = 0$, for any measurable $\lambda \in [0,1]$, the pair (λ, h_0) is a solution of Model 2. Hence, Model 2 is ill-posed.

Example 4. Let us consider

$$\Omega =]0,1[, \quad E = 0, \quad f = 0, \quad h_0 = \sum_{i=1}^{n-1} \alpha_i I_{[a_i,a_{i+1}[} + \alpha_n I_{[a_{n-1},a_n]}, \quad \alpha_i > 0.$$

Then:

1) for any continuous λ such that $\lambda(\alpha_i) = 0$ $(i = 1,\ldots,n)$, the pair (λ, h_0) is a solution of Model 2;

2) if the solution of Model 1 exists, then, with allowance for the previous remarks, necessarily, $\lambda = 1$ and $1 \in \Lambda_{\rm ad}$. Hence, there exists h^* such that the pair $(1, h^*)$ is a solution of Model 1, i.e., $\Delta h_0 \geqslant 0$ in $D'(\Omega)$ [by virtue of weak continuity of Δh_0 at $t = 0$ and continuity of the derivative $\partial_t h$ in $D'(\Omega)$]. Such a condition is impossible (Δh_0 is not a Radon measure), and Model 1 is ill-posed.

It should be noted that the research is performed under the assumption of compatible and rather smooth data.

3.2. *Inverse Problem.* Another important problem in geology is the inverse problem: For a given topography h, is it possible to find an appropriate multiplier–limiter λ? The problem is then stated as follows: determine the function λ satisfying the equation

$$\operatorname{div}(\lambda\nabla h) = \partial_t h \quad \text{in} \quad Q$$

and the boundary conditions

$$-\lambda\partial_n h = f \quad \text{on} \quad]0,T[\times\Gamma_s, \qquad \lambda\partial_n h + f \geqslant 0, \quad (\lambda\partial_n h + f)(\partial_t h + E) = 0 \quad \text{on} \quad]0,T[\times\Gamma_e.$$

One has to find $\lambda = \lambda(h)$ such that the limiter λ is maximum for a given h, i.e.,

$$1-\lambda \geqslant 0, \quad \partial_t h + E \geqslant 0, \quad (1-\lambda)(\partial_t h + E) = 0 \quad \text{in} \quad Q.$$

3.3. *Conclusions about This Modeling.* The examples considered show that Models 1 and 2 in the general case are ill-posed (the solution is either nonexistent or nonunique). These models can be well-posed only for a special family of initial conditions.

One has to formulate a correct direct problem before resolving the inverse one.

4. New Approach to Model 2. Let us introduce a new way to consider Model 2. We denote by H the maximum monotone graph of the Heaviside function, i.e., $H(x) = 0$ if $x < 0$, $H(x) = 1$ if $x > 0$, and $H(0) \in [0,1]$. Then, formally, the pair (λ, h) is a solution of Model 2 if $\lambda \in H(\partial_t h + E)$ in Q and one has

$$\partial_t h - \operatorname{div}(\lambda\nabla h) = 0 \quad \text{in } Q \tag{7}$$

with appropriate boundary and initial conditions. Indeed,

1. By construction, (λ, h) is a solution of (1)–(3) and (5).

2. If $\partial_t h + E < 0$ in $\omega \subset Q$, $\omega \neq \emptyset$, then $\lambda = 0$ and $\partial_t h = 0$ in ω. As E is nonnegative, $\partial_t h + E$ is also nonnegative in ω, i.e., one has a contradiction. Thus, $\partial_t h + E \geqslant 0$ in $Q \setminus Q_0$, and $Q_0 = \emptyset$.

3. If $\lambda \in [0,1]$ and $\lambda < 1$, then $\partial_t h + E \leqslant 0$. Thus, $(1-\lambda)(\partial_t h + E) = 0$ in $Q \setminus Q_0$, and $Q_0 = \emptyset$.

We begin with approximation of the Heaviside function H by a piecewise-linear continuous function H_ε and consider the equation

$$\partial_t h_\varepsilon - \operatorname{div}(H_\varepsilon(\partial_t h_\varepsilon + E)\nabla h_\varepsilon) = 0 \quad \text{in } Q$$

with appropriate boundary and initial conditions. In what follows, the subscript ε is omitted for convenience.

4.1. *Remark on the Type of the Regularized Equation.* Let us consider the equation

$$\partial_t u - \operatorname{div}(\lambda(\partial_t u)\nabla u) = 0,$$

which is a hyperbolic degenerating one. In fact, differentiating this equation, we obtain

$$\partial_t u = \operatorname{div}(\lambda(\partial_t u)\nabla u) = \lambda(\partial_t u)\Delta u + \lambda'(\partial_t u)\nabla\partial_t u\nabla u.$$

In the case of one spatial variable for $\Omega =]0,1[$, the discriminant of this equation is determined by the formula

$$\Delta_u = -|\lambda'(\partial_t u)\partial_x u|^2/4 \leqslant 0.$$

Introducing the new function

$$w = \lambda(\partial_t u)\partial_x u, \qquad \partial_x w = \partial_x(\lambda(\partial_t u)\partial_x u) = \partial_t u,$$

we obtain the degenerating hyperbolic equation

$$\lambda(\partial_x w)\partial_t w = \lambda'(\partial_x w)\partial_{xt} w + \lambda^2(\partial_x w)\partial_{xx} w$$

with the discriminant $\Delta_w = -|\lambda'(\partial_x w)|^2/4 \leqslant 0$. Note, the equation of this type describes a one-dimensional unsteady vertical filtration flow in inhomogeneous multi-stratum soil (see [8, 9]). In this case, the pressure $\psi(z,t)$ and moisture $\theta(\psi)$ can be found from the equation

$$\partial_t\theta - \partial_z(K(\theta,z)(\partial_z\psi - 1)) = f(\theta,z,t) = 0 \quad \text{in} \quad Q = \Omega\times]0,T[. \tag{8}$$

The hydraulic conductivity is usually found by the following formula:

$$K(\theta,z) = \begin{cases} K_s((\theta-\theta_r)/(\theta_s-\theta_r))^n, & \theta_r < \theta \leqslant \theta_s, \\ 0, & \theta \leqslant \theta_r. \end{cases}$$

Here K_s, θ_r, θ_s, and $n > 1$ are constants. The function $f(\theta,z,t)$ determines the intensity of extraction of soil water. The dependence of moisture θ on pressure ψ with allowance for its hysteresis is determined as a certain modification of Gardner's formula. For $\psi \geqslant 0$, we obtain $\theta = \theta_s$, and for $\psi < 0$, we have

$$\theta = \theta_0 + (\theta_s - \theta_0)(1 + (1-p)|a\psi|^m)/(1 + |a\psi|^m), \qquad \partial_t\psi > 0,$$

$$\theta = \theta_0 - p(\theta_s - \theta_0)/(1 + |b\psi|^m), \qquad \partial_t\psi \leqslant 0.$$

Here, the parameter p is constant within the range of t where $\partial_t\psi$ retains its sign ($-1 \leqslant p < 0$ for $\partial_t\psi \leqslant 0$ and $0 \leqslant p \leqslant 1$ for $\partial_t\psi$). At points where the derivative $\partial_t\psi$ changes its sign, the new value of the parameter p is determined from the condition of continuity for ψ and θ

$$\psi\Big|_{t\pm 0} = \psi\Big|_{t\mp 0}, \qquad -p/(1+|b\psi|^m)\Big|_{t\pm 0} = (1+(1-p)|a\psi|^m)/(1+|a\psi|^m)\Big|_{t\mp 0}$$

for $\partial_t\psi\Big|_{t\pm 0} \leqslant 0$ and $\partial_t\psi\Big|_{t\mp 0} > 0$. At the initial time $t = 0$, the distribution $p(z)$ should be given. The parameters K_s, θ_r, θ_s, θ_0, n, m, a, and b depend on the soil type. Under the conditions mentioned above, Eq. (8) can be represented as

$$\partial_t u - \partial_z(\lambda(z,u,u_t)\partial_z u) = \tilde{f} \quad \text{in} \quad Q = \Omega\times]0,T[,$$

where $u = \theta$ and $\tilde{f} = f + \partial_z K(\theta, z)$.

The hyperbolicity of the considered equation $\partial_t u - \operatorname{div}(\lambda(\partial_t u)\nabla u) = 0$ justifies the search for the solution by using the technique of traveling waves.

4.2. *Solutions of the Traveling Wave Type.* Let us consider the conditions

$$\Omega =]0,1[, \qquad \Gamma_e = \{0\}, \qquad \Gamma_s = \{1\},$$

$$E^* > E \geqslant 0, \qquad \xi = \mu x + t \quad (\mu > 0), \qquad 0 < \xi_0 < \xi_1,$$

$$E(\xi) = EI_{[0,\xi_0[}(\xi) + E^* I_{]\xi_0,+\infty]}(\xi),$$

and look for the special solution

$$h(t,x) = h(\xi), \qquad \lambda(t,x) = \lambda(\xi).$$

Example 5. For $(\xi_1 - \xi_0)/\mu^2 + E/E^* \leqslant 1$, the pair (h,λ) determined by the equations

$$h(x,t) = \mu^2 E\,\mathrm{e}^{-\xi_0/\mu^2}[1 - \mathrm{e}^{\xi/\mu^2}] + h_0(0), \qquad \lambda(x,t) = 1, \qquad 0 < \xi \leqslant \xi_0,$$

$$h(x,t) = E^*(\xi_0 - \xi) + h_0(0) - \mu^2 E(1 - \mathrm{e}^{-\xi_0/\mu^2}), \qquad \lambda(x,t) = (\xi - \xi_0)/\mu^2 + E/E^*, \qquad \xi_0 < \xi \leqslant \xi_1$$

is a solution of Model 2.

Example 6. For $E \leqslant E^*\,\mathrm{e}^{(\xi_0-\xi_1)/\mu^2}$, the pair (h,λ) determined by the equations

$$h(x,t) = -E\xi + h_0(0), \quad \lambda(x,t) = (\xi - \xi_0)/\mu^2 + 1, \qquad 0 < \xi \leqslant \xi_0,$$

$$h(x,t) = \mu^2 E(1 - \mathrm{e}^{(\xi-\xi_0)/\mu^2}) - E\xi_0 + h_0(0), \qquad \lambda(x,t) = 1, \qquad \xi_0 \leqslant \xi \leqslant \xi_1$$

is a solution of Model 2.

Example 7. Let us consider the variable $\xi = x + \mu t$. Then, if $\xi_1 = \xi_0 + \min\{\mu(h_0(0) - E/\mu^2)/E^*, (E^* - E)/(\mu E^*)\}$, the pair (h,λ) determined by the equations

$$h(x,t) = E\,\mathrm{e}^{-\mu\xi_0}(1 - \mathrm{e}^{\mu\xi})/\mu^2 + h_0(0), \qquad \lambda(x,t) = 1, \qquad 0 < \xi \leqslant \xi_0,$$

$$h(x,t) = E^*(\xi_0 - \xi)/\mu + h_0(0) - E(1 - \mathrm{e}^{-\mu\xi_0})/\mu^2, \qquad \lambda(x,t) = \mu(\xi - \xi_0) + E/E^*, \qquad \xi_0 \leqslant \xi \leqslant \xi_1$$

is a solution of Model 2.

The construction of these examples' calculus and some graphics can be found in [4].

5. New Conservation Law. Approximation Problems. We study the simplified conservation laws

$$\partial_t u - \operatorname{div}(H(\partial_t u)\nabla u) \ni 0 \quad \text{or} \quad \partial_t u - \operatorname{div}(H_\varepsilon(\partial_t u)\nabla u) = 0 \quad \text{in } Q,$$

where H is the maximum monotone graph of the Heaviside function in the case $E = 0$ and H_ε is a suitable regularization of H.

5.1. *Preparatory Approximation Problem.* Let us consider the following problem: find a pair (χ, U) such that

$$\chi \in L^\infty(Q), \qquad \chi \in H(\partial_t u),$$

$$\partial_t u - \operatorname{div}(\chi\nabla u) = 0 \qquad \text{in} \quad Q,$$

$$\chi\partial_n u + f = 0 \qquad \text{on} \quad]0,T[\times\Gamma_e,$$

$$\chi\partial_n u + f \geqslant 0, \qquad \partial_t u \geqslant 0, \qquad (\chi\partial_n u + f)\partial_t u = 0 \quad \text{on} \quad]0,T[\times\Gamma_s,$$

$$u(0,x) = u_0(x) \qquad \text{in} \quad \Omega.$$

Here f is nonpositive on $]0,T[\times\Gamma_s$ and $u_0 \in H^1(\Omega)\cap L^\infty(\Omega)$ for $u_0 \geqslant 0$. In this section, we offer some mathematical tools for obtaining an ultraweak solution. Such solutions are the limit in $C([0,T],L^2(\Omega))$ of approximating solutions obtained from implicit time discretization schemes [10]. Hence, approximation formulas are derived for numerical methods.

Let ε and h be positive real numbers. For an arbitrary x, we introduce the functions

$$H_\varepsilon(x) = \max[\varepsilon, \min(x/\varepsilon) + \varepsilon, 1], \qquad F_\varepsilon(x) = \int\limits_0^x \frac{1}{H_\varepsilon(t)}\,dt.$$

Then, we can prove the following statement.

Proposition 3. *There exists a unique sequence $u_\varepsilon^k \in H^1(\Omega)$ such that $u_\varepsilon^0 = u_0$ and, $\forall v \in H^1(\Omega)$, $v \geqslant 0$ on Γ_s, the following inequality is satisfied:*

$$\int\limits_\Omega \frac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h}\Big(v - \frac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h}\Big)\,dx + \int\limits_\Omega H_\varepsilon\Big(\frac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h}\Big)\nabla u_\varepsilon^k \cdot \nabla\Big(v - \frac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h}\Big)\,dx$$

$$\leqslant \int\limits_{\Gamma_s} f_k\Big(v - \frac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h}\Big)\,d\sigma, \qquad f_k = \frac{1}{h}\int\limits_{kh}^{(k+1)h} f(s)\,ds.$$

Moreover, $0 \leqslant u_\varepsilon^k \leqslant \sup\limits_\Omega u_0$ in Ω and $u_\varepsilon^k \geqslant u_\varepsilon^{k-1}$ on Γ_s.

Proof. The inequality $0 \leqslant u_\varepsilon^k \leqslant \sup\limits_\Omega u_0$ follows from the maximum principle. To prove the existence of u_ε^k, one has to use the Schauder–Tikhonov fixed-point theorem in an adapted penalization of the constraint $u_\varepsilon^k \geqslant u_\varepsilon^{k-1}$ on Γ_s. The uniqueness follows from the known methods of contracting semigroup techniques.

5.2. *Some Estimates of the Solutions.* Based on the above-mentioned properties, we come to the following statements.

Lemma 1. *The sequence u_ε^k is bounded in $L^\infty(\Omega)$, independently of ε and h.*

Lemma 2. *For any integer N, the following estimate is valid:*

$$\frac{2}{h}\sum_{k=1}^N \|u_\varepsilon^k - u_\varepsilon^{k-1}\|^2_{L^2(\Omega)} + \|\nabla u_\varepsilon^N\|^2_{(L^2(\Omega))^n} + \sum_{k=1}^N \|\nabla(u_\varepsilon^k - u_\varepsilon^{k-1})\|^2_{(L^2(\Omega))^n} \leqslant \|\nabla u_0\|^2_{(L^2(\Omega))^n}.$$

Proof. To prove Lemmas 1 and 2, one has to use the function $v = (u_\varepsilon^k - u_\varepsilon^{k-1})/h - \varepsilon F_\varepsilon(u_\varepsilon^k - u_\varepsilon^{k-1})/h$ as a test function and note that

$$h\int\limits_\Omega (u_\varepsilon^k - u_\varepsilon^{k-1})F_\varepsilon\Big(\frac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h}\Big)\,dx \geqslant \|u_\varepsilon^k - u_\varepsilon^{k-1}\|^2_{L^2(\Omega)},$$

$$h\int\limits_\Omega H_\varepsilon\Big(\frac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h}\Big)\nabla u_\varepsilon^k \cdot \nabla F_\varepsilon\Big(\frac{u_\varepsilon^k - u_\varepsilon^{k-1}}{h}\Big)\,dx = \int\limits_\Omega \nabla u_\varepsilon^k \cdot \nabla u_\varepsilon^k - u_\varepsilon^{k-1}\,dx.$$

We introduce the function

$$\hat{u}_{\varepsilon,h}(t,x) = \begin{cases} (u_\varepsilon^k - u_\varepsilon^{k-1})(t-kh)/h + u_\varepsilon^{k-1}, & t \in [kh;(k+1)h], \\ u_0(x), & t \in [0;h]. \end{cases}$$

Then, one can prove the following proposition.

Proposition 4. *Independently of ε and h, the sequence $\hat{u}_{\varepsilon,h}$ is bounded in $H^1(Q)\cap L^\infty(Q)\cap L^\infty(0,T;H^1(\Omega))$. Then, the sequence $\hat{u}_{\varepsilon,h}$ is relatively compact in $C([0,T],L^2(\Omega))$ by Ascoli's lemma.*

5.3. *Limit of the Sequence as $h \to O$ for a Fixed ε.* Note, $\forall v \in L^2(0,T;H^1(\Omega))$ and $v \geqslant 0$ on $]0,T[\times\Gamma_s$, the following inequality is valid:

$$\int\limits_Q \partial_t\hat{u}_{\varepsilon,h} v\,dx\,dt + \int\limits_Q a_\varepsilon(\partial_t\hat{u}_{\varepsilon,h})\nabla\hat{u}_{\varepsilon,h}\cdot\nabla v\,dx\,dt \geqslant \int\limits_{]0,T[\times\Gamma_s} fv\,d\sigma\,dt + O(h).$$

The presence of two weak converging terms $a_\varepsilon(\partial_t \hat{u}_{\varepsilon,h})$ and $\nabla \hat{u}_{\varepsilon,h}$ prevents from passing to the limit over the time step $h \to O$ and from obtaining a solution of the partial differential equation in the sense of distributions. To give a limit formulation to this inequality, one can use the Young measure theory. Let us recall the properties of vectorial Young measures suggested in [11].

Proposition 5. *Let Q be a bounded open set $\mathbb{R}^d$ and $\boldsymbol{u}_n$ be a bounded sequence in $[L^2(Q)]^d$. Then, there exists a vectorial Young measure ν in $L^\infty_w(Q,\mathbb{R}^d)$ such that*

$$\forall f \in C(\mathbb{R}^d,\mathbb{R}) \quad \exists c > 0 \quad \text{and} \quad \forall \xi \in \mathbb{R}^d \quad |f(\boldsymbol{u})| \leqslant c(1+|\xi|),$$

hence,

$$\forall v \in L^2(Q) \qquad \int\limits_Q f(\boldsymbol{u}_n) v \, dx \to \int\limits_{Q\times\mathbb{R}^d} f(\xi) v(x) \, d\nu_x(\xi) \, dx.$$

Since $0 \leqslant a_\varepsilon \leqslant 1$, in accordance with the previous *a priori* estimates, these exist u^ε in $H^1(Q)$ and a vectorial Young measure ν^ε in $L^\infty_w(Q,\mathbb{R}^{N+1})$ such that

$$\partial_t u^\varepsilon(t,x) = \int\limits_{\mathbb{R}^{N+1}} \xi_0 \, d\nu^\varepsilon_{(t,x)}(\xi), \qquad \partial_{x_i} u^\varepsilon(t,x) = \int\limits_{\mathbb{R}^{N+1}} \xi_i \, d\nu_{(t,x)}\varepsilon(\xi),$$

where $\xi = (\xi_0, \xi_1, \ldots, \xi_N) \in \mathbb{R}^{N+1}$, and the following inequality is satisfied:

$$\int\limits_Q \partial_t u^\varepsilon v \, dx \, dt + \sum_{i=1}^N \int\limits_{Q\times\mathbb{R}}^{N+1} a_\varepsilon(\xi_0)\xi_i \, \partial_{x_i} v \, d\nu^\varepsilon_{(t,x)}(\xi) \, dx \, dt \geqslant \int\limits_{]0,T[\times\Gamma_s} f v \, d\sigma \, dt$$

$\forall v \in L^2(0,T;H^1(\Omega))$ and $v \geqslant 0$ on $]0,T[\times\Gamma_s$. In particular, it follows from here that

$$\partial_t u^\varepsilon - \operatorname{div}\Big[\int\limits_{\mathbb{R}^{N+1}} a_\varepsilon(\xi_0)\xi_i \, d\nu^\varepsilon_{(t,x)}(\xi)\Big] = 0.$$

5.4. *Limit as $\varepsilon \to O$ for a Fixed h.* We assume that h is fixed. Then, the following proposition is valid:

Proposition 6. *There exists a paired sequence u^k, χ_k in $H^1(\Omega) \times L^\infty(\Omega)$ such that $u^0_\varepsilon = u_0$, $\chi_k \in H((u^k - u^{k-1})/h))$ and, $\forall v \in H^1(\Omega)$ and $v \geqslant 0$ on Γ_s, the following inequality is satisfied:*

$$\int\limits_\Omega \frac{u^k - u^{k-1}}{h}\Big(v - \frac{u^k - u^{k-1}}{h}\Big) dx + \int\limits_\Omega \chi_k \nabla u^k \cdot \nabla\Big(v - \frac{u^k - u^{k-1}}{h}\Big) dx \geqslant \int\limits_{\Gamma_s} f_k\Big(v - \frac{u^k - u^{k-1}}{h}\Big) d\sigma.$$

Moreover, $0 \leqslant u^k \leqslant \sup\limits_\Omega u_0$ in Ω and $u^k \geqslant u^{k-1}$ on Γ_s.

Proof. This result comes inductively from the previous *a priori* estimates and the identity

$$\begin{aligned} H_\varepsilon\Big(\frac{u^k_\varepsilon - u^{k-1}}{h}\Big)\nabla u^k_\varepsilon &= h H_\varepsilon\Big(\frac{u^k_\varepsilon - u^{k-1}}{h}\Big)\nabla \frac{u^k_\varepsilon - u^{k-1}}{h} - H_\varepsilon\Big(\frac{u^k_\varepsilon - u^{k-1}}{h}\Big)\nabla u^{k-1} \\ &= h\nabla A_\varepsilon\Big(\frac{u^k_\varepsilon - u^{k-1}}{h}\Big) - H_\varepsilon\Big(\frac{u^k_\varepsilon - u^{k-1}}{h}\Big)\nabla u^{k-1}, \end{aligned}$$

where $A'_\varepsilon = H_\varepsilon$.

We introduce the function

$$\hat{u}_h(t,x) = \begin{cases} (u^k - u^{k-1})(t-kh)/h + u^{k-1}, & t \in [kh;(k+1)h], \\ u_0(x), & t \in [0;h]. \end{cases}$$

Then, the following proposition is valid.

Proposition 7. *Independently of the time step h, the sequence $\hat{u}_h$ is bounded in $H^1(Q) \cap L^\infty(Q) \cap L^\infty(0,T;H^1(\Omega))$ and, hence, relatively compact in $C([0,T],L^2(\Omega))$. Moreover, $\chi_h = \sum\limits_{k\geqslant 0} \chi_k I_{[kh,(k+1)h[} \in H(\partial_t \hat{u}_h)$, $\partial_t \hat{u}_h \geqslant 0$ in Q.*

The results obtained above can also be proved for the following problem: find (χ, u) such that

$$\chi \in L^\infty(Q), \qquad \chi \in H(\partial_t u + E),$$

$$\partial_t u - \operatorname{div}(\chi \nabla u) = 0 \qquad \text{in } Q,$$

$$\partial_n u = 0 \quad \text{on} \quad]0,T[\times\Gamma_e, \qquad u = 0 \quad \text{on} \quad]0,T[\times\Gamma_s,$$

$$u(0,x) = u_0(x) \quad \text{in } \Omega,$$

where E is a nonnegative function of the variable t and $u_0 \geqslant 0$ belongs to $H^1(\Omega) \cap L^\infty(\Omega)$.

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